EXTENDING PROPERTIES TO RELATIVELY HYPERBOLIC GROUPS

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Abstract. Consider a finitely generated group G that is relatively hyperbolic with respect to a family of subgroups $H_1, \ldots, H_n$. We present an axiomatic approach to the problem of extending metric properties from the subgroups H_i to the full group G. In particular, we show that both *finite decomposition complexity* and *straight finite decomposition complexity* are extendable properties, as are certain weakened versions.

1. Introduction

The concept of relative hyperbolicity was proposed by Gromov in [9], as a generalization of hyperbolicity. Farb, Bowditch, Osin, and Mineyev–Yemen, [1, 6, 16, 14], have developed this in various directions, which are equivalent for finitely generated groups. We follow the approach to relatively hyperbolicity given by Osin [16].

Say G is a finitely generated group that is relatively hyperbolic with respect to a family of subgroups $\{H_i\}_{i=1}^n$, as defined in Section 2. Various authors have considered the problem of extending metric properties of the subgroups H_i to the full group G. In particular, *finite asymptotic dimension*, *coarse embeddability* (also known as *uniform embeddability*), and *exactness* are all known to be extendable [15, 3, 17]. The main goal of this article is to show that *finite decomposition complexity* and *straight finite decomposition complexity* are extendable properties.

Finite decomposition complexity (FDC) was introduced in [10] as a natural generalization of finite asymptotic dimension, and was used to study rigidity properties of manifolds. The more general notion of straight finite decomposition complexity (sFDC) was recently introduced in [5]. (We review the definitions in Section 4.) The class of groups with FDC is already quite large, and contains all countable linear groups [10, Theorem 3.1]. By [5, Theorem 3.4], all metric spaces with sFDC satisfy Yu's Property A, so finitely generated groups with sFDC satisfy the coarse Baum–Connes conjecture [20].

In this article, we present an axiomatic approach to the problem of extendability. Consider a property $\mathcal{P}$ of *metric families* (that is, sets of metric spaces). We say that a metric space X has the property $\mathcal{P}$ if the metric family $\{X\}$ has $\mathcal{P}$. We will focus attention on properties that are coarsely invariant, in the sense that if a metric space X has $\mathcal{P}$, then so do all metric spaces Y that are coarsely equivalent to X. For such properties $\mathcal{P}$, we say a group G has $\mathcal{P}$ if it has $\mathcal{P}$ when equipped with a proper, left-invariant metric.

We identify several conditions that such a property $\mathcal{P}$ may satisfy, which together imply the extendability of $\mathcal{P}$ for relatively hyperbolic groups. These conditions

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are Coarse Inheritance, the Finite Union Theorem, the Union Theorem, and the Transitive Fibering Theorem, which are defined in Section 3. We also assume that $\mathcal{P}$ is satisfied by all metric spaces with finite asymptotic dimension. Our main tool for extending such properties is the work of Osin [15] regarding the *relative Cayley graph* of a relatively hyperbolic group.

These results interact nicely with recent work in algebraic K–theory. It was shown in [18] that the Integral K–theoretic Novikov Conjecture (injectivity of the K–theoretic assembly map) holds for all group rings $R[G]$, where R is a unital ring and G has finite decomposition complexity and a finite classifying space $K(G,1)$. Goldfarb [8], building on joint work with Carlsson [2], has shown that in fact the assembly map is an *isomorphism* under these conditions. If G is torsion-free and relatively hyperbolic with respect to subgroups $\{H_i\}_{i=1}^n$ satisfying the conditions of these theorems, then G also satisfies the conditions: by Corollary 3.11, G has finite decomposition complexity, and by [7, Theorem A.1], there exists a finite $K(G,1)$[1].

Goldfarb's work relies on a proof that finitely generated groups with sFDC satisfy *weak regular coherence*, which guarantees the existence of projective resolutions of finite length for certain $R[\Gamma]$–modules over sufficiently well-behaved coefficient rings R. A simple modification to Goldfarb's argument (see Remark 4.15) shows that sFDC can be replaced by the weakened version introduced in Section 4, and we show that weak sFDC is also extendable for relatively hyperbolic groups.

2. Relatively Hyperbolic Groups

Suppose G is a finitely generated group with a finite symmetric generating set S, and let $\{H_i\}_{i=1}^k$ be a family of finitely generated subgroups. Then G is a quotient of the free product $F = F(S) * H_1 * H_2 * \cdots * H_k$, where $F(S)$ is the free group on S. Say that G is *finitely presented relative to* $\{H_i\}_{i=1}^k$ if the kernel of the projection $F \to G$ is the normal closure of a finite subset $\mathcal{R}$ in F. (Note that if G is finitely presented, then it is also finitely presented relative to $\{H_i\}_{i=1}^k$).

Set $\mathcal{H} = \sqcup_{i=1}^k (H_i \setminus \{1\})$. If a word w in the alphabet $S \cup \mathcal{H}$ represents the identity element of G, it can be expressed in the form $w = \prod_{j=1}^m a_i^{-1} r_i^{\pm 1} a_i$ where $r_i \in \mathcal{R}$ and $a_i \in F$ for $i = 1, \ldots, m$. The smallest possible number m in such a representation of w is the *relative area* of w, denoted by $Area_{rel}(w)$.

Definition 2.1. *G is* hyperbolic relative to the collection of subgroups $\{H_i\}_{i=1}^k$ *if it is finitely presented relative to $\{H_i\}_{i=1}^k$ and there is a constant K such that every word w in $S \cup \mathcal{H}$ that represents the identity in G satisfies $Area_{rel}(w) \leq K\|w\|$, where $\|w\|$ represents the length of the word in $S \cup \mathcal{H}$.*

A key construction in relatively hyperbolic groups is the relative Cayley graph, $\Gamma(G, S \cup \mathcal{H})$; that is, the Cayley graph of G with respect to the generating set $S \cup \mathcal{H}$. This graph is not locally finite. However Osin has proven the following.

Theorem 2.2 ([15, Theorem 17])**.** *The relative Cayley graph $\Gamma(G, S \cup \mathcal{H})$ has finite asymptotic dimension.*

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[1] Kasprowski [13] has shown that the Integral K–theoretic Novikov Conjecture holds for all groups G with finite decomposition complexity and a *finite-dimensional* classifying space. Hence it would be interesting to know whether the property of having a finite-dimensional classifying space is extendable (at least to torsion-free groups).

The existence of constants L and ε involved in the following two lemmas (from [15]) will be necessary in what follows, though the results themselves will not be mentioned again. The terminology and notation is taken from [15].

Lemma 2.3. *Suppose that a group G is generated by a finite set S and is hyperbolic relative to $\{H_i\}_{i=1}^k$. Then there is a constant $L > 0$ such that for every cycle q in $\Gamma(G, S \cup \mathcal{H})$, every $i \in \{1, \dots, k\}$, and every set of isolated H_i-components $p_1, \dots, p_m$ of q, we have*

$$\sum_{j=1}^{m} d_S\left((p_j)_-, (p_j)_+\right) \leq L\|q\|.$$

Lemma 2.4. *Suppose that a group G is generated by a finite set S and is hyperbolic relative to $\{H_i\}_{i=1}^k$. Then for any $s \geq 0$, there is a constant $\varepsilon = \varepsilon(s) \geq 0$ such that the following condition holds. Let p_1 and p_2 be two geodesics in $\Gamma(G, S \cup \mathcal{H})$ such that $d_S\left((p_1)_-, (p_1)_+\right) \leq s$ and $d_S\left((p_2)_-, (p_2)_+\right) \leq s$. Let c be a component of p_1 such that $d_S\left(c_-, c_+\right) \geq \varepsilon$. Then there is a component of p_2 connected to c.*

3. Extendable properties

Many properties can be extended from the peripheral subgroups $H_1, \dots, H_n$ to the group G. Coarse embeddability [3], exactness [17], finite asymptotic dimension [15], and combability [12] are just a few examples of such properties. An analysis of [3] and [15] shows much similarity in method.

Suppose that $\mathcal{P}$ is some property of metric families. We isolate a few features that may hold for $\mathcal{P}$, which will be of interest. Recall that a map between metric spaces, $f\colon X \to Y$, is *uniformly expansive* if there exists a nondecreasing function $\rho\colon [0, \infty) \to [0, \infty)$ such that for all $x, x' \in X$, $d_Y(f(x), f(x')) \leq \rho(d_X(x, x'))$. Such a map is *homogeneous* if for all $y_1, y_2 \in \operatorname{im}(f) \subset Y$ there exist isometries $\phi\colon X \to X$ and $\bar{\phi}\colon Y \to Y$ such that

- $f \circ \phi = \bar{\phi} \circ f$, and
- $\bar{\phi}(y_1) = y_2$.

Lemma 3.1. *Let G be a finitely generated group, with finite symmetric generating set S, and let $\mathcal{H}$ be a finite family of subgroups. Then the map $p\colon G \to \Gamma(G, S \cup \mathcal{H})$, which sends a group element to the vertex it represents, is homogeneous.*

Proof. Let $g, g' \in G$. Denote by v_g and $v_{g'}$ the vertices in $\Gamma(G, S \cup \mathcal{H})$ identified with g and g', respectively. As p is equivariant with respect to the group action, we define $\phi\colon G \to G$ and $\bar{\phi}\colon \Gamma(G, S \cup \mathcal{H}) \to \Gamma(G, S \cup \mathcal{H})$ through left multiplication by the element $g'g^{-1}$. Thus $\bar{\phi}\,(g) = g'$, and $p \circ \phi = \bar{\phi} \circ p$. □

There are several versions of the Fibering Theorem. We will establish the following version for straight finite decomposition complexity in Section 4. Recall that we say a metric space X has $\mathcal{P}$ if the family $\{X\}$ has $\mathcal{P}$.

Definition 3.2 (Homogeneous Fibering Theorem)**.** *Say that $\mathcal{P}$* satisfies the Homogeneous Fibering Theorem *if the following holds.*

> *Let $f\colon E \to B$ be a uniformly expansive, homogeneous map. Assume B has property $\mathcal{P}$ and for each bounded subset $D \subset B$, the inverse image $f^{-1}(D)$ has property $\mathcal{P}$. Then E has property $\mathcal{P}$.*

A significantly weaker version of the above will suffice for studying extendability. We say that a map $f\colon X \to Y$ of metric spaces is *contractive*, or a *contraction*, if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Such maps are uniformly expansive.

Definition 3.3 (Transitive Fibering Theorem)**.** *Say that* P satisfies the Transitive Fibering Theorem *if the following holds.*

> *Let* Γ *be a countable group acting isometrically on* E *and* B*, and assume* B *has finite asymptotic dimension and that* Γ *acts transitively on* B*. Let* $f : E \to B$ *be a contractive,* Γ*–equivariant map. If for each bounded subset* $D \subset B$*,* $f^{-1}(D)$ *has property* $\mathcal{P}$*, then* E *has property* $\mathcal{P}$*.*

We note that the maps p considered in the Transitive Fibering Theorem are automatically homogeneous, since Γ is acting by isometries.

Definition 3.4 (Finite Union Theorem)**.** *Say that* $\mathcal{P}$ satisfies the Finite Union Theorem *if the following holds.*

> *Let* X *be a metric space written as a finite union of metric subspaces* $X = \cup_{i=1}^{n} X_i$*. If each* X_i *has* $\mathcal{P}$ *then so does* X*.*

The next property addresses more general unions. Recall that two subsets A, B of a metric space X are said to be r–disjoint if $d(A, B) > r$.

Definition 3.5 (Union Theorem)**.** *Say that* $\mathcal{P}$ satisfies the Union Theorem *if the following holds.*

> *Let* X *be a metric space written as a union of metric subspaces* $X = \cup_{i\in\mathcal{I}} X_i$*. Suppose that* $\{X_i\}_{i\in\mathcal{I}}$ *has* $\mathcal{P}$ *and that for every* $r > 0$ *there exists a metric subspace* $Y(r) \subset X$ *with* $\mathcal{P}$ *such that the sets* $Z_i(r) = X_i \setminus Y(r)$ *are pairwise* r*–disjoint. Then* X *has* $\mathcal{P}$*.*

Definition 3.6 (Coarse Inheritance)**.** *Say that* $\mathcal{P}$ satisfies Coarse Inheritance *if the following holds.*

> *Let* X *and* Y *be metric spaces. If there is a coarse embedding from* X *to* Y *and* Y *has* $\mathcal{P}$*, then so does* X*.*

Note that if $\mathcal{P}$ satisfies Coarse Inheritance, then it is a coarsely invariant property.

Definition 3.7. *Say that* $\mathcal{P}$ *is* axiomatically extendable *if it satisfies the Transitive Fibering Theorem, the Finite Union Theorem, the Union Theorem, and Coarse Inheritance, and every metric space with finite asymptotic dimension has* $\mathcal{P}$*.*

Proposition 3.8. *Coarse embeddability, exactness, and finite decomposition complexity* (*see Definition 4.4*) *are axiomatically extendable properties.*

Coarse embeddability and exactness for metric families are defined in [3, Definitions 2.2 and 2.8], where they are referred to as 'equi-embeddability' and 'equi-exactness'.

Proof. For coarse embeddability, the Coarse Inheritance property is clear. The Finite Union Theorem and the Union Theorem are Corollaries 4.5 and 4.6 of [3]. The Transitive Fibering Theorem is a special case of Corollary 4.7 of [3]. Finally, spaces of finite asymptotic dimension are coarsely embeddable [19].

We now turn to exactness. Again, the Coarse Inheritance property follows easily from the definition. Metric spaces of finite asymptotic dimension are exact, by

Proposition 4.3 of [3]. The Finite Union Theorem, Union Theorem, and Transitive Fibering Theorem come from Corollaries 4.5, 4.6, and 3.4 of [3].

For finite decomposition complexity, Coarse Inheritance, the Finite Union Theorem, the Union Theorem, and a stronger version of the Fibering Theorem appear in Section 3.1 of [11]. That spaces of finite asymptotic dimension have finite decomposition complexity is proven in [11, Section 4], using [4]. □

Theorem 3.9. *Suppose that $\mathcal{P}$ is an axiomatically extendable property. If G is relatively hyperbolic with respect to $H_1, \ldots, H_n$ and each H_i has $\mathcal{P}$, then G has $\mathcal{P}$.*

We begin by proving an auxiliary lemma. Let

$$B(n) = \{g \in G \,:\, d_{S\cup\mathcal{H}}(e,g) \leq n\}\,.$$

That is, $B(n)$ is the closed ball around e of radius n in $\Gamma(G, S\cup\mathcal{H})$. We consider $B(n)$ as a metric subspace of G, with the word metric associated to S.

Lemma 3.10. *Suppose that each H_i has $\mathcal{P}$. For any integer $n > 0$, $B(n)$ has $\mathcal{P}$.*

Note that H_i having $\mathcal{P}$ with respect to a word metric d_{H_i} associated to a finite generating set of H_i is equivalent to H_i having $\mathcal{P}$ with respect to the restricted metric from G, since both are proper left-invariant metrics and $\mathcal{P}$ is coarsely invariant.

Proof. The argument is based on the proof of [15, Lemma 3.2]. Proceed by induction on n. For $n = 1$, $B(1) = S \cup \left(\cup_{i=1}^{k} H_i\right)$ has $\mathcal{P}$ by the Finite Union Theorem. Let $n > 1$ and assume $B(m)$ has $\mathcal{P}$ for all positive integers $m < n$. We have

$$B(n) = \left(\bigcup_{i=1}^{k} B(n-1)H_i\right) \cup \left(\bigcup_{s\in S} B(n-1)s\right).$$

As each $B(n-1)s$ is coarsely equivalent to $B(n-1)$ and S is finite, $\bigcup_{s\in S} B(n-1)s$ has $\mathcal{P}$ by the Finite Union Theorem and the induction hypothesis. It remains to check that $\bigcup_{i=1}^{k} B(n-1)H_i$ has $\mathcal{P}$.

Fix $i \in \{1, \ldots, k\}$ and let $R(n-1)$ be a subset of $B(n-1)$ such that

$$B(n-1)H_i = \bigsqcup_{r\in R(n-1)} rH_i.$$

Fix an $s > 0$ and set

$$T_s = \{g \in G \,:\, d_S(e,g) \leq \max\{\varepsilon, 2L(s+1)\}\}\,,$$

where L and $\varepsilon = \varepsilon(s)$ are the constants from Lemmas 2.3 and 2.4 respectively. Let $Y_s = B(n-1)T_s$. As T_s is finite, Y_s has $\mathcal{P}$. Osin shows in [15, Lemma 3.2] that the sets $\{rH_i \setminus Y_s \,:\, r \in R(n-1)\}$ are s–disjoint, so $B(n-1)H_i$ has $\mathcal{P}$ by the Union Theorem. The Finite Union Theorem then shows $\bigcup_{i=1}^{k} B(n-1)H_i$ has $\mathcal{P}$. □

Proof of Theorem 3.9. Consider the map $p\colon G \to \Gamma(G, S\cup\mathcal{H})$. This is a contraction, thus it is uniformly expansive. By Theorem 2.2, $\Gamma(G, S\cup\mathcal{H})$ has finite asymptotic dimension, so $\Gamma(G, S\cup\mathcal{H})$ has the property $\mathcal{P}$ as well.

For each bounded subset Z of $\Gamma(G, S\cup\mathcal{H})$ there is an n such that $p^{-1}(Z)$ lies in $B(n)$. As $B(n)$ has $\mathcal{P}$, $p^{-1}(Z)$ has $\mathcal{P}$ as well by Coarse Inheritance. Consider the map $p\colon G \to \mathrm{Image}(p)$, which is equivariant with respect to the transitive left-translation actions of G (in fact, p is simply the identity map on underlying set G). Since $\Gamma(G, S\cup\mathcal{H})$ has finite asymptotic dimension, so does $\mathrm{Im}(p) \subset \Gamma(G, S\cup\mathcal{H})$. By the Transitive Fibering Theorem, G has the property $\mathcal{P}$. □

Corollary 3.11. *Suppose G is relatively hyperbolic with respect to $H_1, \dots, H_n$. If each H_i has finite decomposition complexity, so does G.*

The same argument shows that this result holds with FDC replaced by either of the weak versions (k–FDC or wFDC) discussed in the next section, since the extendability arguments for FDC in [11] all apply to these weak versions as well.

4. Extendability of straight finite decomposition complexity

We recall the definition of finite decomposition complexity from [11].

Definition 4.1. *An* (k, r)–decomposition *of a metric space X over a metric family $\mathcal{Y}$ is a decomposition*

$$X = X_0 \cup X_1 \cup \cdots \cup X_{k-1}, \quad X_i = \bigsqcup_{r-disjoint} X_{ij},$$

where each $X_{ij} \in \mathcal{Y}$. A metric family $\mathcal{X}$ is (k, r)–decomposable *over $\mathcal{Y}$ if every member of $\mathcal{X}$ admits a (k, r)–decomposition over $\mathcal{Y}$.*

When $k = 2$, we recover the notion of r–decomposition from [11].

Remark 4.2. *If X admits a (k, r)–decomposition over a metric family $\mathcal{Y}$, then it also admits a (k', r)–decomposition over $\mathcal{Y}$ for each $k' \geqslant k$, since we may repeat the spaces X_i appearing in the decomposition.*

Definition 4.3. *Let $\mathfrak{U}$ be a collection of metric families. A metric family $\mathcal{X}$ is k–*decomposable *over $\mathfrak{U}$ if, for every $r > 0$, there is a metric family $\mathcal{Y} \in \mathfrak{U}$ and a (k, r)–decomposition of $\mathcal{X}$ over $\mathcal{Y}$. The collection $\mathfrak{U}$ is* stable under k–fold decomposition *if every metric family which k–decomposes over $\mathfrak{U}$ actually belongs to $\mathfrak{U}$.*

A metric family is weakly decomposable *over $\mathfrak{U}$ if it is k–decomposable over $\mathfrak{U}$ for some $k \in \mathbb{N}$.*

Recall that a metric family $\mathcal{Z}$ is *uniformly bounded* if

$$\sup\{\mathrm{diam}(Z) : Z \in \mathcal{Z}\} < \infty.$$

Definition 4.4. *The collection $\mathfrak{D}^k$ of metric families with k–fold finite decomposition complexity (k–FDC) is the smallest collection of metric families that contains the uniformly bounded metric families and is stable under k–fold decomposition. When $k = 2$, we recover the notion of FDC from* [10, 11].

The collection $w\mathfrak{D}$ of metric families with weak finite decomposition complexity (wFDC) is the smallest collection of metric families that contains the uniformly bounded metric families and is stable under weak decomposition.

By Remark 4.2, we have $\mathfrak{D}^1 \subset \mathfrak{D}^2 \subset \mathfrak{D}^3 \subset \cdots \subset \mathrm{w}\mathfrak{D}$.

Remark 4.5. *As explained in* [11] *and in* [18, Section 6] *for the case of FDC, the collections $\mathfrak{D}^k$ may be defined as unions of collections of families $\mathfrak{D}^k_\alpha$ indexed by (countable) ordinals, by starting with $\mathfrak{D}^k_0 = \mathcal{B}$, the collection of uniformly bounded metric families, and then inductively defining $\mathfrak{D}^k_{\alpha+1}$ to be the set of metric families that k–decompose over $\mathfrak{D}^k_\alpha$ (for limit ordinals β, one may simply set $\mathfrak{D}^k_\beta = \bigcup_{\alpha<\beta} \mathfrak{F}_\alpha$). The same remark applies to $w\mathfrak{D}$.*

In [5], Dranishnikov and Zarichnyi give the following generalization of FDC, whose applications to algebraic K–theory have been studied by Goldfarb [8]. Here we extend this notion somewhat by considering weak versions.

Definition 4.6. *A metric family $\mathcal{X}$ has* straight finite decomposition complexity (*sFDC*) *if, for every sequence $R_1 < R_2 < \ldots$ of positive numbers, there exists an $n \in \mathbb{N}$ and metric families $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$, the family $\mathcal{X}_i$ is R_{i+1}–decomposable over $\mathcal{X}_{i+1}$, and the family $\mathcal{X}_n$ is uniformly bounded.*

Metric families with k–fold straight finite decomposition complexity (*k–sFDC*) *are defined analogously, by replacing R_i–decomposability by (k, R_i)–decomposability. Note that 2–sFDC is the same as sFDC.*

A metric family $\mathcal{X}$ has weak straight finite decomposition complexity *with respect to the sequence $\mathbf{k} = (k_1, k_2, \ldots)$ ($k_i \in \mathbb{N}$) if for every sequence $R_1 < R_2 < \ldots$ of positive numbers, there exists an $n \in \mathbb{N}$ and metric families $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$, the family $\mathcal{X}_i$ is (k_{i+1}, R_{i+1})–decomposable over $\mathcal{X}_{i+1}$, and the family $\mathcal{X}_n$ is uniformly bounded. We say that $\mathcal{X}$ has weak straight finite decomposition complexity (wsFDC) if it has wsFDC with respect to some sequence $(k_1, k_2, \ldots)$.*

Note that a metric family $\mathcal{X}$ has k–sFDC if and only if it has wsFDC with respect to the constant sequence $k, k, k, \ldots$. Also, by Remark 4.2 every space (or family) with wsFDC actually has wsFDC with respect to a non-decreasing sequence $(k_1, k_2, \ldots)$, because we may always replace k_i by $\max\{k_1, \ldots, k_i\}$.

We have the following diagram of implications relating these concepts ($k \geqslant 2$):

$$\begin{array}{ccccccc}
\text{FAD} & \Longrightarrow & \text{FDC} & \Longrightarrow & k\text{–FDC} & \Longrightarrow & \text{wFDC} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{sFDC} & \Longrightarrow & k\text{–sFDC} & \Longrightarrow & \text{wsFDC}
\end{array}$$

In particular, spaces of finite asymptotic dimension have all of the above properties.

By [5, Theorem 3.4], all metric spaces with sFDC satisfy Yu's Property A, and the proof extends easily to show that for each k, spaces with k–sFDC have Property A. Additionally, it is shown in [11, Theorem 4.3] that all bounded geometry metric spaces with weak FDC have Property A. However, we do not know how to extend these arguments to spaces having just weak sFDC.

Question 4.7. *Do all* (*bounded geometry*) *spaces with wsFDC satisfy Property A?*

Remark 4.8. *One may imagine a further weakening of wsFDC: a metric family $\mathcal{X}$ has "very weak" sFDC if for each sequence $0 < R_1 < R_2 < \ldots$, there exists an $n \in \mathbb{N}$, a finite sequence $k_1, k_2, \ldots, k_n$, and metric families $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_n$ such that $\mathcal{X} = \mathcal{X}_0$, the family $\mathcal{X}_i$ is (k_{i+1}, R_{i+1})–decomposable over $\mathcal{X}_{i+1}$, and the family $\mathcal{X}_n$ is uniformly bounded. However, every discrete metric space with bounded geometry has this property; so in particular this property does not imply Property A.*

We now establish basic extendability properties for weak versions of straight finite decomposition complexity. The usual argument for coarse inheritance of FDC also proves the following result.

Lemma 4.9. *If X has wsFDC with respect to the sequence $\mathbf{k} = (k_1, k_2, \ldots)$ and there exists a coarse embedding $Y \to X$, then Y also has wsFDC with respect to $\mathbf{k}$. In particular, the properties k–sFDC and wsFDC satisfy coarse inheritance.*

For the next result, the following notion for metric families will be useful.

Definition 4.10. *Let $\mathcal{X}$ be a metric family. The* subspace closure *of $\mathcal{X}$, denoted by $\mathcal{X}'$ is the metric family $\mathcal{X}' = \{X : \text{ there exists } Y \in \mathcal{X} \text{ with } X \subset Y\}$.*

Theorem 4.11. *Let $f\colon E \to B$ be a uniformly expansive, homogeneous map. Assume that B has wsFDC with respect to a non-decreasing sequence $\mathbf{k} = (k_1, k_2, \ldots)$, and assume that there exists $b_0 \in B$ and a non-decreasing sequence $\mathbf{k}' = (k_1', k_2', \ldots)$ such that for each $r > 0$, the space $f^{-1}(B_r(b_0))$ has wsFDC with respect to $\mathbf{k}'$. Then E has wsFDC with respect to the sequence $(k_1'', k_2'', \ldots)$, where $k_i'' = \max\{k_i, k_i'\}$.*

In particular, k–sFDC satisfies the Homogeneous Fibering Theorem ($k \geqslant 1$).

Proof. Replacing k_i and k_i' by $\max\{k_i, k_i'\}$, we may assume that B and $f^{-1}(B_r(b_0))$ have wsFDC with respect to the same non-decreasing sequence $\mathbf{k} = (k_1, k_2, \ldots)$.

Take ρ to be the function from the definition of uniform expansion for f, and let $R_1 < R_2 < \ldots$ be given. Since B has wsFDC with respect to $\mathbf{k}$, there is an $n \in \mathbb{N}$ and a sequence of metric families $\mathcal{Y}_0 = \{B\}$, $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ such that $\mathcal{Y}_{i-1}$ is $(k_i, \rho(R_i))$–decomposable over $\mathcal{Y}_i$ and $\mathcal{Y}_n$ is a uniformly bounded family. Let

$$f^{-1}(\mathcal{Y}_i) = \left\{ f^{-1}(Y) \,:\, Y \in \mathcal{Y}_i \right\}.$$

Then $f^{-1}(\mathcal{Y}_0) = \{E\}$, and $f^{-1}(\mathcal{Y}_i)$ can be (k_{i+1}, R_{i+1})–decomposed over $f^{-1}(\mathcal{Y}_{i+1})$, since inverse images of $\rho(R_{i+1})$–disjoint sets in B are R_{i+1}–disjoint in E.

This yields a sequence of decompositions of E that ends with the family $f^{-1}(\mathcal{Y}_n)$, and by assumption there exists $r > 0$ such that each $Y \in \mathcal{Y}_n$ has diameter at most r. Each $f^{-1}(Y)$ is isometric, via one of the isometries $\bar{\phi}$ guaranteed by the definition of homogeneity, to a subspace of $f^{-1}(B_r(b_0))$, so by Lemma 4.9 we conclude that each space $f^{-1}(Y)$ has wsFDC with respect to $\mathbf{k}$.

Applying the definition of wsFDC to the space $f^{-1}(B_r(b_0))$ and the sequence of numbers $R_{n+1} < R_{n+2} < \cdots$ shows that there exists $N \geq 0$ and metric families

$$\mathcal{Z}_n(b_0) = \left\{ f^{-1}(B_r(b_0)) \right\}, \mathcal{Z}_{n+1}(b_0), \mathcal{Z}_{n+2}(b_0), \ldots, \mathcal{Z}_{n+N}(b_0)$$

such that $\mathcal{Z}_{n+N}(b_0)$ is uniformly bounded and for $i = 0, \ldots N-1$, $\mathcal{Z}_{n+i}(b_0)$ admits a (k_{n+i+1}, R_{n+i+1})–decomposition over $\mathcal{Z}_{n+i+1}(b_0)$ (since $k_{n+i+1} \geqslant k_1$).

For $i = 0, \ldots, N$, let $\mathcal{Z}_{n+i}$ be the union over $b \in B$ of all translates of spaces in $\mathcal{Z}_{n+i}(b_0)$ under the isometries $\bar{\phi}$. Since decomposability is defined element-wise over elements in a metric family, we see that $\mathcal{Z}_{n+i}$ admits a (k_{n+i+1}, R_{n+i+1})–decomposition over $\mathcal{Z}_{n+i+1}$. Let $\mathcal{Z}_{n+i}'$ be the subspace closure of $\mathcal{Z}_{n+i}$, and note that $\mathcal{Z}_{n+N}'$ is still uniformly bounded. If $Z' \subset Z$ are metric spaces, then each decomposition of Z can be intersected with Z' to obtain a decomposition of Z'. Hence $\mathcal{Z}_{n+i}'$ admits a (k_{n+i+1}, R_{n+i+1})–decomposition over $\mathcal{Z}_{n+i+1}'$, and the same idea shows that $f^{-1}(\mathcal{Y}_n)$ admits a (k_{n+1}, R_{n+1})–decomposition over $\mathcal{Z}_{n+1}'$.

The sequence of families

$$f^{-1}(\mathcal{Y}_1), f^{-1}(\mathcal{Y}_2), \ldots, f^{-1}(\mathcal{Y}_n), \mathcal{Z}_{n+1}', \ldots, \mathcal{Z}_{n+N}'$$

shows that E has wsFDC with respect to $\mathbf{k}$. □

The Finite Union Theorem for sFDC was established in [5, Theorem 3.5]. We extend this to wsFDC via a somewhat different argument.

Lemma 4.12. *Let $X = \bigcup_{i=1}^N X_i$ and assume that X_j has wsFDC with respect to a non-decreasing sequence $(k_{j1}, k_{j2}, \ldots)$. Then X has wsFDC with respect to the sequence $(k_1, k_2, \ldots)$, where $k_i = \max\{k_{1i}, \ldots, k_{ni}, 2\}$. Hence wsFDC and k–sFDC ($k \geqslant 2$) satisfy the Finite Union Theorem.*

Proof. Say $N = 2$ and let $R_1 < R_2 < \cdots$ be given. Then $X = X_1 \cup X_2$ is (trivially) a (k_1, R_1)–decomposition of X over the family $\mathcal{X}_1 = \{X_1, X_2\}$. Applying the definition of wsFDC to the sequence $R_2 < R_3 < \cdots$ shows that there exist metric families $\mathcal{X}_{1i}$ and $\mathcal{X}_{2i}$, $i = 2, \ldots n$ and (k_{i+1}, R_{i+1})–decompositions of $\mathcal{X}_{ji}$ over $\mathcal{X}_{ji+1}$ $(j = 1, 2)$, with $\mathcal{X}_{1n}$ and $\mathcal{X}_{2n}$ uniformly bounded; note that we are using the assumption $k_i \leqslant k_{i+1}$. Setting $\mathcal{X}_i = \mathcal{X}_{1i} \cup \mathcal{X}_{2i}$ $(i = 2, \ldots, n)$ completes the proof for $N = 2$, and the full result follows by induction on N. □

The proof of the Union Theorem for sFDC given in [5, Theorem 3.6] immediately generalizes to prove the following result.

Lemma 4.13. *Let $X = \bigcup_{i\in\mathcal{I}} X_i$ be a metric space and assume the family $\{X_i\}_{i\in\mathcal{I}}$ has wsFDC with respect to a non-decreasing sequence $\mathbf{k} = (k_1, k_2, \ldots)$. If there exists a non-decreasing sequence $\mathbf{k}' = (k_1', k_2', \ldots)$ such that for each $r > 0$ there exists a subspace $Y(r) \subset X$ such that $Y(r)$ has wsFDC with respect to $\mathbf{k}'$ and $\{\mathcal{X}_i \backslash Y(r)\}_{i\in\mathcal{I}}$ is pairwise r–disjoint, then X has wsFDC with respect to* $\max\{k_i, k_i'\}$.

In particular, k–sFDC satisfies the Union Theorem.

Corollary 4.14. *For each $k \geqslant 2$, k–fold straight finite decomposition complexity is an axiomatically extendable property. In particular if G is relatively hyperbolic with respect to $H_1, \ldots, H_n$ and each H_i has sFDC, then G has sFDC.*

Moreover, if G is relatively hyperbolic with respect to $H_1, \ldots, H_n$ and each H_i has wsFDC, then G has wsFDC.

Proof. We have shown that k–sFDC satisfies Coarse Inheritance, the Union and Finite Union Theorems, and the Homogeneous Fibering Theorem, hence also the Transitive Fibering Theorem, so the result for k–sFDC follows from Theorem 3.9.

The proof for wsFDC is similar. We adopt the terminology and notation of Section 3. If each H_i has wsFDC, then they all have wsFDC with respect to some common non-decreasing sequence $\mathbf{k} = (k_1, k_2, \ldots)$ with $k_i \geqslant 2$ for each i. The proof of Lemma 3.10, along with Lemmas 4.12 and 4.13, shows that for each $n > 0$, $B(n)$ has wsFDC with respect to $\mathbf{k}$. Consider the homogeneous, uniformly expansive map $G \to \Gamma(G, S \cup \mathcal{H})$ as in the proof of Theorem 3.9. The base has FAD and hence also wsFDC with respect to $\mathbf{k}$ (since $k_i \geqslant 2$), and the bounded fibers all lie inside $B(n)$ for some n. Lemma 4.9 and Theorem 4.11 complete the proof. □

Remark 4.15. *Goldfarb* [8] *studied modules over group rings $R[\Gamma]$ when Γ has sFDC and established results about projective resolutions that play a key role in Carlsson and Goldfarb's work on the K–theoretic Borel conjecture* [2]. *Goldfarb's results all extend to groups with wsFDC. The key geometric component of* [8] (*and the only place where sFDC is needed*) *is the proof of* [8, Theorem 2.5].

We briefly explain how to adapt Goldfarb's argument. We will assume the terminology and notation from [8]. *It suffices to show that if Γ has wsFDC, then the kernel of a b–bicontrolled map $f\colon F \to G$ of Γ–modules, where F is D–lean and G is d–insular, is finitely generated. Goldfarb shows that $K = \ker(f)$ is D'–split, where $D' = D + 2b + d$. If Γ has wsFDC with respect to $(k_1, k_2, \ldots)$, then setting*

$$R_i = 2D + 2b + 2d + 2\left(\left(\sum_{l=1}^{i-1} k_l - (i-1)\right) D' + (i-1)D\right),$$

there exist metric families $\mathcal{Y}_0 = \{\Gamma\}, \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ such that for each i, $\mathcal{Y}_i$ admits a (k_{i+1}, R_{i+1}) decomposition over $\mathcal{Y}_{i+1}$. For a subspace $Y \subset \Gamma$, let $Y[r] = \{x \in \Gamma : d(x, Y) < r\}$. Goldfarb's argument shows that each $x \in K$ is a finite sum of elements in the finitely generated submodules $K\left(Y\left[\sum_{i=1}^{n} k_i - n + nD\right]\right)$, with

$Y \in \mathcal{Y}_n$. The only necessary modification to Goldfarb's argument is the observation that if $X = X_0 \cup \cdots \cup X_{k_1} \subset \Gamma$, and $x \in K(X)$, then splitness of K lets us write $x = x_1 + x_1'$ with $x_1 \in K(X_0[D'])$ and $x_1' \in K(X_1[D'] \cup \cdots \cup X_{k_1}[D'])$, and continuing inductively, x_1' can be written as $x_2 + x_3 + \cdots + x_{k_1}$, where $x_i \in K(X_i[iD'])$ (and in fact $x_{k_1} \in K(X_{k_1}[(k_1 - 1)D'])$. Our choice of R_i guarantees that the decompositions above remain well-separated even after applying these thickenings.

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